

SOME CONDITIONS IMPLYING NORMALITY OF OPERATORS

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ABSTRACT. Let $T \in \mathbb{B}(\mathcal{H})$ and $T = U|T|$ be its polar decomposition. We proved that (i) if T is log-hyponormal or p -hyponormal and $U^n = U^*$ for some n , then T is normal; (ii) if the spectrum of U is contained in some open semicircle, then T is normal if and only if so is its Aluthge transform $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$.

1. INTRODUCTION

Let $\mathbb{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space \mathcal{H} with the identity I . A subspace $\mathcal{K} \subseteq \mathcal{H}$ is said to reduce $T \in \mathbb{B}(\mathcal{H})$ if both $T\mathcal{K} \subseteq \mathcal{K}$ and $T^*\mathcal{K} \subseteq \mathcal{K}$ hold. We say that an operator T is p -hyponormal for some $p > 0$ if $(T^*T)^p \geq (TT^*)^p$. If $p = 1$, T is said to be hyponormal. Clearly T is hyponormal if and only if $\|T\xi\| \geq \|T^*\xi\|$ for any $\xi \in \mathcal{H}$. If T is an invertible operator satisfying $\log(T^*T) \geq \log(TT^*)$, then it is called log-hyponormal, see [13].

Let $T = U|T|$ be the polar decomposition of T , where $\ker(U) = \ker(|T|)$ and U^*U is the projection onto $\text{ran}(|T|)$. It is known that if T is invertible, then U is unitary and $|T|$ is also invertible. It is easy to see that

$$|T^*|^s = U|T|^sU^* \quad (1)$$

for every nonnegative number s . If T is invertible, then

$$\log |T^*| = U(\log |T|)U^*. \quad (2)$$

The Aluthge transformation \tilde{T} of T is defined by $\tilde{T} := |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. This notion was first introduced by Aluthge [1] and is a powerful tool in the operator theory. The reader is referred to [7] for undefined notions and terminology.

One interesting problem in the operator theory is to investigate some conditions under which certain operators are normal. Several mathematicians have paid attention to this problem, see [1, 2, 6, 8] and references therein. One of interesting articles, which presents some results about this topic is that of Stampfli [11]. He showed, among other things, that for a hyponormal operator A , if A^n is normal for some positive integer n , then A is normal. The problem had already been considered in the case when $n = 2$ by Putnam [9]. The results were generalized later to the other classes of operators by a number of authors, for instance, Embry [5], Radjavi and Rosenthal [10] and Duggal [4]. There is another point of view about this issue via spectrum $\text{sp}(\cdot)$. In [11] it is proved that if the spectrum of a hyponormal operator contains only a finite number of limited points or has zero area, then the operator is normal. Using Aluthge transform, this aspect is generalized to p -hyponormal and log-hyponormal operators. In fact, if T is p -hyponormal or log-hyponormal, then \tilde{T} is hyponormal [7, Theorem 1.3.4.1 and

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Theorem 2.3.4.2]. Due to $\text{sp}(A) = \text{sp}(\tilde{A})$ [2, Corollary 2.3], \tilde{A} is normal. Now the result is concluded from the fact that \tilde{A} is normal if and only if so is A [14, Lemma 3]. There are some applications of the subject in other areas of the operator theory that was a motivation for our work, see [8].

In this paper we present some new conditions under which certain operators are normal. We also use a Fuglede–Putnam commutativity type theorem to show that an invertible operator $T = U|T|$, where $\text{sp}(U)$ is contained in an open semicircle, is normal if and only if so is \tilde{T} .

2. MAIN RESULTS

We start this section with one of our main result.

Theorem 2.1. *Let $T \in \mathbb{B}(\mathcal{H})$ be log-hyponormal or p -hyponormal and $T = U|T|$ be the polar decomposition of T such that $U^{n_0} = U^*$ for some positive integer n_0 . Then T is normal.*

Proof. Assume that T is p -hyponormal for some $p > 0$. Hence $|T|^{2p} \geq |T^*|^{2p} = U|T|^{2p}U^*$ by (1). By multiplying both sides of this inequality by U and U^* we have $U|T|^{2p}U^* \geq U^2|T|^{2p}U^{2*}$ whence $|T|^{2p} \geq U|T|^{2p}U^* \geq U^2|T|^{2p}U^{2*}$. By repeating this process, we reach the following sequence of operator inequalities:

$$|T|^{2p} \geq |T^*|^{2p} = U|T|^{2p}U^* \geq U^2|T|^{2p}U^{2*} \geq \dots \geq U^{n_0+1}|T|^{2p}U^{(n_0+1)*} \geq \dots \quad (3)$$

Because of $U^{n_0} = U^*$ we can observe that $U^{n_0+1} = U^*U = U^{(n_0+1)*}$ is the projection onto $\overline{\text{ran}(|T|)}$. Hence $U^{n_0+1}|T|^{2p}U^{(n_0+1)*} = |T|^{2p}$, from which and inequalities (3) we obtain $|T|^{2p} = |T^*|^{2p}$. Hence $|T|^2 = |T^*|^2$, i.e., T is normal as desired.

In the case that T is a log-hyponormal operator inequalities (3) are replaced by the inequalities

$$\begin{aligned} \log |T| \geq \log |T^*| &= U(\log |T|)U^* \geq U^2(\log |T|)U^{2*} \\ &\geq \dots \geq U^{n_0+1}(\log |T|)U^{(n_0+1)*} \geq \dots \end{aligned}$$

and the rest of the proof is similar to argument above. \square

We will need the following lemma in the sequel. One can easily prove it by using the fact that $\log(cT) = (\log c)I + \log T$.

Lemma 2.2. *If T and S are two invertible positive operators such that $\log T \geq \log S$ and c is a positive number, then $\log(cT) \geq \log(cS)$.*

Theorem 2.3. *Let $T \in \mathbb{B}(\mathcal{H})$ be log-hyponormal or p -hyponormal and $T = U|T|$ be the polar decomposition of T such that $U^{*n} \rightarrow I$ or $U^n \rightarrow I$ as $n \rightarrow \infty$, where limits are taken in the strong operator topology. Then T is normal.*

Proof. We assume that $U^{*n}\xi \rightarrow \xi$ as $n \rightarrow \infty$ for all $\xi \in \mathcal{H}$. In the case $U^n \rightarrow I$ in the strong operator topology a similar argument can be used. Let T be p -hyponormal and $\xi \in \mathcal{H}$. It follows from (3) that

$$\| |T|^p \xi \| \geq \| |T^*|^p \xi \| = \| |T|^p U^{*n} \xi \| \geq \| |T|^p U^{2*n} \xi \| \geq \dots \geq \| |T|^p U^{*n} \xi \| \geq \dots \quad (4)$$

Since

$$| \| |T|^p U^{*n} \xi \| - \| |T|^p \xi \| | \leq \| |T|^p U^{*n} \xi - |T|^p \xi \| \leq \| |T|^p \| \| U^{*n} \xi - \xi \| \rightarrow 0$$

as $n \rightarrow \infty$, we have $\| |T|^p U^{*n} \xi \| \rightarrow \| |T|^p \xi \|$ as $n \rightarrow \infty$. Hence, by (4) we get $\| |T|^p \xi \|^2 = \| |T^*|^p \xi \|^2$, so $|T|^{2p} = |T^*|^{2p}$. Thus T is normal.

Now let T be a log-hyponormal operator. Since T is invertible there exists $c > 0$ such that $c|T^*| \geq I$, so $\log(c|T^*|) \geq 0$. Due to $\log |T| \geq \log |T^*| = U(\log |T|)U^*$ we have $\log(c|T|) \geq \log(c|T^*|) = U \log(c|T|)U^*$ by Lemma 2.2 and equality (2). The rest of the proof is similar to the argument above and the proof of Theorem 2.1 so we omit it. \square

In the sequel we are going to present a relationship between an operator and its Aluthge transform. We essentially apply the following lemma.

Lemma 2.4. [12] *Let $T, S \in \mathbb{B}(\mathcal{H})$. Then the following assertions are equivalent:*

- (i) *If $TX = XS$, then $T^*X = XS^*$ for any $X \in \mathbb{B}(\mathcal{H})$.*
- (ii) *If $TX = XS$ where $X \in \mathbb{B}(\mathcal{H})$, then $\overline{R(X)}$ reduces T , $(\ker X)^\perp$ reduces S , and operators $T|_{\overline{R(X)}}$ and $S|_{(\ker X)^\perp}$ are normal.*

Theorem 2.5. *Let $T \in \mathbb{B}(\mathcal{H})$ be an invertible operator and $T = U|T|$ be the polar decomposition of T . Let $\text{sp}(U)$ be contained in some open semicircle. Then \tilde{T} is normal if and only if so is T .*

Proof. Assume that \tilde{T} is normal. Hence $\tilde{T}X = X\tilde{T}$ implies $\tilde{T}^*X = X\tilde{T}^*$ for any $X \in \mathbb{B}(\mathcal{H})$ by Fuglede–Putnam commutativity theorem. We first show that $TX = XT$ implies $T^*X = XT^*$ for any $X \in \mathbb{B}(\mathcal{H})$. Let $X \in \mathbb{B}(\mathcal{H})$ and $TX = XT$. Then $U|T|X = XU|T|$, whence

$$\begin{aligned} \tilde{T}(|T|^{\frac{1}{2}}X|T|^{\frac{-1}{2}}) &= |T|^{\frac{1}{2}}(U|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}X)|T|^{\frac{-1}{2}} \\ &= |T|^{\frac{1}{2}}(X|T|^{\frac{-1}{2}}|T|^{\frac{1}{2}}U|T|)|T|^{\frac{-1}{2}} \\ &= (|T|^{\frac{1}{2}}X|T|^{\frac{-1}{2}})\tilde{T}. \end{aligned} \tag{5}$$

By (5) and the assumption with $|T|^{\frac{1}{2}}X|T|^{\frac{-1}{2}}$ instead of X we have

$$\begin{aligned} |T|^{\frac{1}{2}}U^*|T|X|T|^{\frac{-1}{2}} &= |T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}(|T|^{\frac{1}{2}}X|T|^{\frac{-1}{2}}) = \tilde{T}^*(|T|^{\frac{1}{2}}X|T|^{\frac{-1}{2}}) \\ &= (|T|^{\frac{1}{2}}X|T|^{\frac{-1}{2}})\tilde{T}^* = |T|^{\frac{1}{2}}X|T|^{\frac{-1}{2}}|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}} \\ &= |T|^{\frac{1}{2}}XU^*|T|^{\frac{1}{2}}. \end{aligned}$$

So that

$$U^*|T|X = XU^*|T| \tag{6}$$

and $|T|X|T|^{-1} = UXU^*$. Therefore

$$|T|X|T|^{-1} = U^*(U|T|X)|T|^{-1} = U^*(XU|T|)|T|^{-1} = U^*XU.$$

Thus $UXU^* = U^*XU$, whence $U^2X = XU^2$.

Now we use Beck and Putnam argument in [3]. We replace U by $e^{\alpha}U$ if it is necessary and assume that $\text{sp}(U)$ is contained in the set $\{e^{i\lambda} : \varepsilon < \lambda < \pi - \varepsilon\}$ for some $\varepsilon > 0$. Let $U = \int_{\varepsilon}^{\pi-\varepsilon} e^{i\lambda} dE(\lambda)$ be the spectral decomposition of U . One has $U^2 = \int_{2\varepsilon}^{2\pi-2\varepsilon} e^{i\lambda} dF(\lambda)$, where $F(\lambda) = E(\frac{\lambda}{2})$. By $U^2X = XU^2$ we have $U^{2n}X = XU^{2n}$ for every $n \in \mathbb{Z}$, so $U^{2n} = \int_{2\varepsilon}^{2\pi-2\varepsilon} e^{in\lambda} dF(\lambda)$. Hence $f(U^2)X = Xf(U^2)$ for every f in the set of all bounded Borel-measurable complex-valued functions on $\{z : |z| = 1\}$ since $\{e^{int}\}$ is complete on the interval $0 \leq t \leq 2\pi$. Hence, by spectral resolution for normal operator U ,

$F(\lambda)X = XF(\lambda)$, whence $E(\lambda)X = XE(\lambda)$ and this implies again that $UX = XU$ and clearly this implies that

$$U^*X = XU^*. \quad (7)$$

From (6) and (7) we obtain

$$|T|X = U(U^*|T|X) = U(XU^*|T|) = U(U^*X)|T| = X|T|. \quad (8)$$

From (7) and (8) we deduce that $T^*X = |T|U^*X = X|T|U^* = XT^*$ as desired. We have shown that $TX = XT$ implies $T^*X = XT^*$ for any $X \in \mathbb{B}(\mathcal{H})$. It follows from Lemma 2.4(ii) for $X = I$ that T is normal.

The reverse is easy. In fact if T is normal, then $\tilde{T} = T$. \square

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